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Comparison of solvers performance when solving the 3D Helmholtz elastic wave equations using the Hybridizable Discontinuous Galerkin method

M. Bonnasse-Gahot^{1,2}, H. Calandra³, J. Diaz¹ and S. Lanteri²

¹ INRIA Bordeaux-Sud-Ouest, team-project Magique 3D

² INRIA Sophia-Antipolis-Méditerranée, team-project Nachos

³ TOTAL Exploration-Production

Motivations

Imaging methods

- ▶ Reverse Time Migration (RTM) : based on the **reversibility of wave equation**
- ▶ Full Wave Inversion (FWI) : **inversion process** requiring to solve **many forward problems**

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Seismic imaging : time-domain or harmonic-domain ?

- ▶ **Time-domain** : **imaging condition complicated** but **quite low computational cost**
- ▶ **Harmonic-domain** : **imaging condition simple** but **huge computational cost**

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Memory usage



Motivations

Resolution of the forward problem of the inversion process

- ▶ Elastic wave propagation in the frequency domain : **Helmholtz equation**

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First order formulation of Helmholtz wave equations

$$\mathbf{x} = (x, y, z) \in \Omega \subset \mathbb{R}^3,$$

$$\begin{cases} i\omega \rho(\mathbf{x}) \mathbf{v}(\mathbf{x}) = \nabla \cdot \underline{\underline{\sigma}}(\mathbf{x}) + \underline{f}_s(\mathbf{x}) \\ i\omega \underline{\underline{\sigma}}(\mathbf{x}) = \underline{\underline{C}}(\mathbf{x}) \underline{\underline{\varepsilon}}(\mathbf{v}(\mathbf{x})) \end{cases}$$

- ▶ \mathbf{v} : velocity vector
- ▶ $\underline{\underline{\sigma}}$: stress tensor
- ▶ $\underline{\underline{\varepsilon}}$: strain tensor

Motivations

Resolution of the forward problem of the inversion process

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- ▶ ρ : mass density
- ▶ $\underline{\underline{C}}$: elasticity tensor
- ▶ f_s : source term, $f_s \in L^2(\Omega)$

Approximation methods

Discontinuous Galerkin Methods

- ✓ unstructured tetrahedral meshes
- ✓ combination between FEM and finite volume method (FVM)
- ✓ *hp*-adaptivity
- ✓ easily parallelizable method

Approximation methods

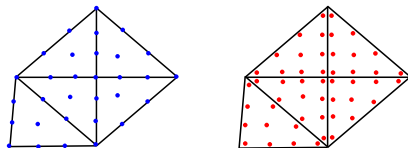
Discontinuous Galerkin Methods

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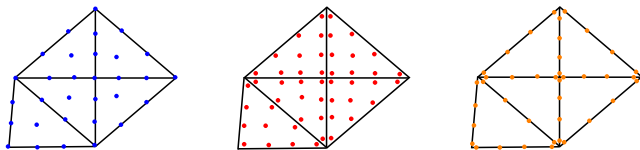
Hybridizable Discontinuous Galerkin Methods

- ✓ same advantages as DG methods : unstructured tetrahedral meshes, hp -adaptivity, easily parallelizable method, discontinuous basis functions
- ✓ introduction of a new variable defined only on the interfaces
- ✓ lower number of coupled DOF than classical DG methods

Approximation methods

Hybridizable Discontinuous Galerkin Methods

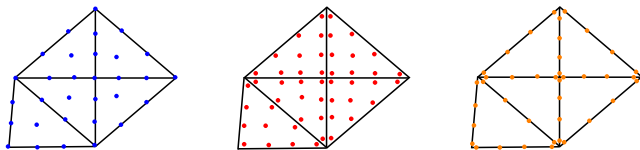
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- ✓ same advantages as DG methods : unstructured tetrahedral meshes, *hp*-adaptivity, easily parallelizable method, discontinuous basis functions
- ✓ introduction of a new variable defined only on the interfaces
- ✓ lower number of coupled DOF than classical DG methods
- ✗ time-domain increases computational costs



Hybridizable Discontinuous Galerkin method

-  B. Cockburn, J. Gopalakrishnan and R. Lazarov. Unified hybridization of discontinuous Galerkin, mixed and continuous Galerkin methods for second order elliptic problems. *SIAM Journal on Numerical Analysis*, Vol. 47 :1319-1365, 2009.
-  S. Lanteri, L. Li and R. Perrussel. Numerical investigation of a high order hybridizable discontinuous Galerkin method for 2d time-harmonic Maxwell's equations. *COMPEL*, 32(3)1112-1138, 2013.
-  N.C. Nguyen, J. Peraire and B. Cockburn. High-order implicit hybridizable discontinuous Galerkin methods for acoustics and elastodynamics. *Journal of Computational Physics*, 230 :7151-7175, 2011
-  N.C. Nguyen and B. Cockburn. Hybridizable discontinuous Galerkin methods for partial differential equations in continuum mechanics. *Journal of Computational Physics* 231 :5955–5988, 2012

Contents

Hybridizable Discontinuous Galerkin method

- Classical HDG Formulation

- Symmetric HDG formulation

- Algorithm

2D Numerical results : comparison of the two HDG formulations

3D numerical results : focus on the resolution part

HDG formulation of the equations

Local HDG formulation

$$\begin{cases} i\omega \rho \mathbf{v} - \nabla \cdot \underline{\underline{\sigma}} &= 0 \\ i\omega \underline{\underline{\sigma}} - \underline{\underline{C}} \varepsilon(\mathbf{v}) &= 0 \end{cases}$$

HDG formulation of the equations

Local HDG formulation

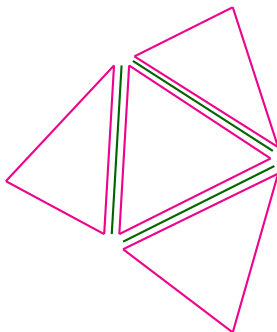
$$\begin{cases} \int_K i\omega \mathbf{v}^K \cdot \mathbf{w} + \int_K \underline{\underline{\sigma}}^K : \nabla \mathbf{w} - \int_{\partial K} \widehat{\underline{\underline{\sigma}}}^{\partial K} \cdot \mathbf{n} \cdot \mathbf{w} = 0 \\ \int_K i\omega \underline{\underline{\sigma}}^K : \underline{\underline{\xi}} + \int_K \mathbf{v}^K \cdot \nabla \cdot (\underline{\underline{c}}^K \underline{\underline{\xi}}) - \int_{\partial K} \widehat{\mathbf{v}}^{\partial K} \cdot \underline{\underline{c}}^K \underline{\underline{\xi}} \cdot \mathbf{n} = 0 \end{cases}$$

$\widehat{\underline{\underline{\sigma}}}^K$ and $\widehat{\mathbf{v}}^K$ are numerical traces of $\underline{\underline{\sigma}}^K$ and \mathbf{v}^K respectively on ∂K

HDG formulation of the equations

We define :

$$\hat{\mathbf{v}}^{\partial K} = \boldsymbol{\lambda}^F, \quad \forall F \in \mathcal{F}_h,$$

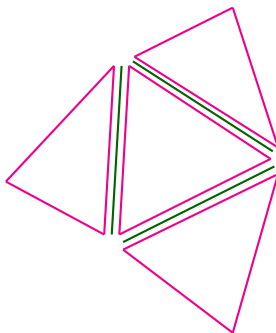


HDG formulation of the equations

We define :

$$\begin{aligned}\hat{\mathbf{v}}^{\partial K} &= \boldsymbol{\lambda}^F, & \forall F \in \mathcal{F}_h, \\ \hat{\underline{\underline{\sigma}}}^{\partial K} \cdot \mathbf{n} &= \underline{\underline{\sigma}}^K \cdot \mathbf{n} - \tau \mathbf{l} (\mathbf{v}^K - \boldsymbol{\lambda}^F), & \text{on } \partial K\end{aligned}$$

where τ is the stabilization parameter ($\tau > 0$)



HDG formulation of the equations

Local HDG formulation

$$\left\{ \begin{array}{l} \int_K i\omega \rho^K \mathbf{v}^K \cdot \mathbf{w} - \int_K (\nabla \cdot \underline{\underline{\sigma}}^K) \cdot \mathbf{w} + \int_{\partial K} \tau \mathbf{l} (\mathbf{v}^K - \lambda^F) \cdot \mathbf{w} = 0 \\ \int_K i\omega \underline{\underline{\sigma}}^K : \underline{\underline{\xi}} + \int_K \mathbf{v}^K \cdot \nabla \cdot (\underline{\underline{C}}^K \underline{\underline{\xi}}) - \int_{\partial K} \lambda^F \cdot \underline{\underline{C}}^K \underline{\underline{\xi}} \cdot \mathbf{n} = 0 \end{array} \right.$$

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We define :

$$\underline{\underline{W}}^K = \left(\underline{\underline{V}}_x^K, \underline{\underline{V}}_y^K, \underline{\underline{V}}_z^K, \underline{\underline{\sigma}}_{xx}^K, \underline{\underline{\sigma}}_{yy}^K, \underline{\underline{\sigma}}_{zz}^K, \underline{\underline{\sigma}}_{xy}^K, \underline{\underline{\sigma}}_{xz}^K, \underline{\underline{\sigma}}_{yz}^K \right)^T$$

$$\underline{\underline{\Lambda}} = \left(\underline{\underline{\Lambda}}^{F_1}, \underline{\underline{\Lambda}}^{F_2}, \dots, \underline{\underline{\Lambda}}^{F_{n_f}} \right)^T, \text{ where } n_f = \text{card}(\mathcal{F}_h)$$

Discretization of the local HDG formulation

$$\underline{\underline{A}}^K \underline{\underline{W}}^K + \sum_{F \in \partial K} \underline{\underline{C}}^{K,F} \underline{\underline{\Lambda}} = 0$$

HDG formulation of the equations

Local HDG formulation

$$\begin{cases} \int_K i\omega \rho^K \mathbf{v}^K \cdot \mathbf{w} - \int_K (\nabla \cdot \underline{\underline{\sigma}}^K) \cdot \mathbf{w} + \int_{\partial K} \tau \mathbf{l} (\mathbf{v}^K - \lambda^F) \cdot \mathbf{w} = 0 \\ \int_K i\omega \underline{\underline{\sigma}}^K : \underline{\underline{\xi}} + \int_K \mathbf{v}^K \cdot \nabla \cdot (\underline{\underline{C}}^K \underline{\underline{\xi}}) - \int_{\partial K} \lambda^F \cdot \underline{\underline{C}}^K \underline{\underline{\xi}} \cdot \mathbf{n} = 0 \end{cases}$$

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Discretization of the local HDG formulation

$$\underline{\underline{A}}^K \underline{\underline{W}}^K + \underline{\underline{C}}^K \underline{\underline{\Lambda}} = 0$$

HDG formulation of the equations

Transmission condition

In order to determine λ^F , the continuity of the normal component of $\underline{\underline{\hat{\sigma}}}^{\partial K}$ is weakly enforced, rendering this numerical trace conservative :

$$\int_F [\![\underline{\underline{\hat{\sigma}}}^{\partial K} \cdot \mathbf{n}]\!] \cdot \boldsymbol{\eta} = 0$$

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Replacing $(\underline{\underline{\hat{\sigma}}}^{\partial K} \cdot \mathbf{n})$ and summing over all faces, the transmission condition becomes :

$$\sum_{K \in \mathcal{T}_h} \int_{\partial K} (\underline{\underline{\hat{\sigma}}}^K \cdot \mathbf{n}) \cdot \eta - \sum_{K \in \mathcal{T}_h} \int_{\partial K} \tau \mathbf{l} (\mathbf{v}^K - \lambda^F) \cdot \eta = 0$$

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Discretization of the transmission condition

$$\sum_{K \in \mathcal{T}_h} [\mathbb{B}^K \underline{\underline{w}}^K + \mathbb{L}^K \underline{\underline{\Lambda}}] = 0$$

HDG formulation of the equations

Global HDG formulation

$$\left\{ \begin{array}{l} \int_K i\omega \rho^K \mathbf{v}^K \cdot \mathbf{w} - \int_K (\nabla \cdot \underline{\underline{\sigma}}^K) \cdot \mathbf{w} + \int_{\partial K} \tau \mathbf{l} (\mathbf{v}^K - \lambda^F) \cdot \mathbf{w} = 0 \\ \int_K i\omega \underline{\underline{\sigma}}^K : \underline{\underline{\xi}} + \int_K \mathbf{v}^K \cdot \nabla \cdot (\underline{\underline{C}}^K \underline{\underline{\xi}}) - \int_{\partial K} \lambda^F \cdot \underline{\underline{C}}_K \underline{\underline{\xi}} \cdot \mathbf{n} = 0 \\ \sum_{K \in \mathcal{T}_h} \int_{\partial K} (\underline{\underline{\sigma}}^K \cdot \mathbf{n}) \cdot \eta - \sum_{K \in \mathcal{T}_h} \int_{\partial K} \tau \mathbf{l} (\mathbf{v}^K - \lambda^F) \cdot \eta = 0 \end{array} \right.$$

HDG formulation of the equations

Global HDG discretization

$$\left\{ \begin{array}{l} \mathbb{A}^K \underline{W}^K + \mathbb{C}^K \underline{\Lambda} = 0 \\ \sum_{K \in \mathcal{T}_h} [\mathbb{B}^K \underline{W}^K + \mathbb{L}^K \underline{\Lambda}] = 0 \end{array} \right.$$

HDG formulation of the equations

Global HDG discretization

$$\left\{ \begin{array}{l} \underline{W}^K = -(\underline{A}^K)^{-1} \underline{C}^K \underline{\Lambda} \\ \sum_{K \in \mathcal{T}_h} [\underline{B}^K \underline{W}^K + \underline{L}^K \underline{\Lambda}] = 0 \end{array} \right.$$

HDG formulation of the equations

Global HDG discretization

$$\sum_{K \in \mathcal{T}_h} [-\mathbf{B}^K (\mathbf{A}^K)^{-1} \mathbf{C}^K + \mathbf{L}^K] \underline{\mathbf{\Lambda}} = 0$$

Symmetric HDG formulation

Local HDG formulation

$$\begin{cases} i\omega\rho\mathbf{v} - \nabla \cdot \underline{\underline{\sigma}} &= 0 \\ i\omega\underline{\underline{\sigma}} - \underline{\underline{C}}\varepsilon(\mathbf{v}) &= 0 \end{cases}$$

Symmetric HDG formulation

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$\underline{\underline{C}}$ invertible and symmetric tensor, i.e for a symmetric $\underline{\underline{\sigma}}$:

$$\underline{\underline{\sigma}} = \underline{\underline{C}}\underline{\underline{\varepsilon}}(\mathbf{u}) \quad \text{and} \quad \underline{\underline{\varepsilon}}(\mathbf{u}) = \underline{\underline{D}}\underline{\underline{\sigma}}$$

with $\underline{\underline{D}} = \underline{\underline{C}}^{-1}$ and $\mathbf{u} = i\omega\mathbf{v}$

Symmetric HDG formulation

Local HDG formulation

$$\begin{cases} \int_K i\omega \rho^K \mathbf{v}^K \cdot \mathbf{w} + \int_K \underline{\underline{\sigma}}^K : \nabla \mathbf{w} - \int_{\partial K} \widehat{\underline{\underline{\sigma}}}^{\partial K} \cdot \mathbf{n} \cdot \mathbf{w} = 0 \\ \int_K i\omega \underline{\underline{\sigma}}^K : \underline{\underline{\xi}} + \int_K \mathbf{v}^K \cdot \nabla \cdot (\underline{\underline{C}}^K \underline{\underline{\xi}}) - \int_{\partial K} \widehat{\mathbf{v}}^{\partial K} \cdot \underline{\underline{C}}^K \underline{\underline{\xi}} \cdot \mathbf{n} = 0 \end{cases}$$

$\widehat{\underline{\underline{\sigma}}}^K$ and $\widehat{\mathbf{v}}^K$ are numerical traces of $\underline{\underline{\sigma}}^K$ and \mathbf{v}^K respectively on ∂K

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$$\underline{\underline{\xi}} = -\underline{\underline{D}}^K \underline{\underline{\xi}}'$$

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Symmetric HDG formulation

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$$\begin{aligned} \hat{\mathbf{v}}^{\partial K} &= \lambda^F, & \forall F \in \mathcal{F}_h, \\ \hat{\underline{\underline{\sigma}}}^{\partial K} \cdot \mathbf{n} &= \underline{\underline{\sigma}}^K \cdot \mathbf{n} - \tau \mathbf{l}(\mathbf{v}^K - \lambda^F), & \text{on } \partial K \end{aligned}$$

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Local HDG formulation

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Discretization of the local HDG formulation

$$\mathbb{A}_2^K \underline{\underline{W}}^K + \sum_{F \in \partial K} \mathbb{C}_2^{K,F} \underline{\underline{\Lambda}} = 0$$

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Discretization of the local HDG formulation

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\mathbb{A}_2^K symmetric matrix

Symmetric HDG formulation

Transmission condition

$$\int_F [[\underline{\hat{\sigma}}^{\partial K} \cdot \mathbf{n}]] \cdot \eta = 0$$

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$$\int_F [[\underline{\hat{\sigma}}^{\partial K} \cdot \mathbf{n}]] \cdot \eta = 0$$

$$\sum_{K \in \mathcal{T}_h} \int_{\partial K} (\underline{\sigma}^K \cdot \mathbf{n}) \cdot \eta - \sum_{K \in \mathcal{T}_h} \int_{\partial K} \tau \mathbf{l} (\mathbf{v}^K - \lambda^F) \cdot \eta = 0$$

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Transmission condition

$$\int_F \llbracket \underline{\underline{\hat{\sigma}}}^{\partial K} \cdot \mathbf{n} \rrbracket \cdot \eta = 0$$

$$\sum_{K \in \mathcal{T}_h} \int_{\partial K} \left(\underline{\underline{\sigma}}^K \cdot \mathbf{n} \right) \cdot \eta - \sum_{K \in \mathcal{T}_h} \int_{\partial K} \tau \mathbf{l} \left(\mathbf{v}^K - \lambda^F \right) \cdot \eta = 0$$

Discretization of the transmission condition

$$\sum_{K \in \mathcal{T}_h} \left[\mathbb{B}^K \underline{\underline{w}}^K + \mathbb{L}^K \underline{\underline{\Lambda}} \right] = 0$$

$$\mathbb{B}^K = (\mathbb{C}_2^K)^T$$

Symmetric HDG formulation

Transmission condition

$$\int_F [[\underline{\hat{\sigma}}^{\partial K} \cdot \mathbf{n}]] \cdot \eta = 0$$

$$\sum_{K \in \mathcal{T}_h} \int_{\partial K} (\underline{\sigma}^K \cdot \mathbf{n}) \cdot \eta - \sum_{K \in \mathcal{T}_h} \int_{\partial K} \tau \mathbf{l} (\mathbf{v}^K - \lambda^F) \cdot \eta = 0$$

Discretization of the transmission condition

$$\sum_{K \in \mathcal{T}_h} [(\mathbf{C}_2^K)^T \underline{w}^K + \mathbb{L}^K \underline{\Lambda}] = 0$$

Symmetric HDG formulation of the equations

Global HDG formulation

$$\left\{ \begin{array}{l} \int_K i\omega \rho^K \mathbf{v}^K \cdot \mathbf{w} - \int_K (\nabla \cdot \underline{\underline{\sigma}}^K) \cdot \mathbf{w} + \int_{\partial K} \tau \mathbf{l} (\mathbf{v}^K - \lambda^F) \cdot \mathbf{w} = 0 \\ - \int_K i\omega \underline{\underline{D}}^K \underline{\underline{\sigma}}^K : \underline{\underline{\xi}}' - \int_K \mathbf{v}^K \cdot \nabla \cdot \underline{\underline{\xi}}' + \int_{\partial K} \lambda^F \cdot \underline{\underline{\xi}}' \cdot \mathbf{n} = 0 \\ \sum_{K \in \mathcal{T}_h} \int_{\partial K} (\underline{\underline{\sigma}}^K \cdot \mathbf{n}) \cdot \eta - \sum_{K \in \mathcal{T}_h} \int_{\partial K} \tau \mathbf{l} (\mathbf{v}^K - \lambda^F) \cdot \eta = 0 \end{array} \right.$$

Symmetric HDG formulation of the equations

Global HDG discretization

$$\left\{ \begin{array}{l} \mathbb{A}_2^K \underline{W}^K + \mathbb{C}_2^K \underline{\Lambda} = 0 \\ \sum_{K \in \mathcal{T}_h} [(\mathbb{C}_2^K)^T \underline{W}^K + \mathbb{L}^K \underline{\Lambda}] = 0 \end{array} \right.$$

Symmetric HDG formulation of the equations

Global HDG discretization

$$\left\{ \begin{array}{l} \underline{W}^K = -(\underline{A}_2^K)^{-1} \underline{C}_2^K \underline{\Lambda} \\ \sum_{K \in \mathcal{T}_h} [(\underline{C}_2^K)^T \underline{W}^K + \underline{L}^K \underline{\Lambda}] = 0 \end{array} \right.$$

Symmetric HDG formulation of the equations

Global HDG discretization

$$\sum_{K \in \mathcal{T}_h} [-(\mathbb{C}_2^K)^T (\mathbb{A}_2^K)^{-1} \mathbb{C}_2^K + \mathbb{L}^K] \underline{\Lambda} = 0$$

Symmetric HDG formulation of the equations

Global HDG discretization

$$\sum_{K \in \mathcal{T}_h} [-(\mathbb{C}_2^K)^T (\mathbb{A}_2^K)^{-1} \mathbb{C}_2^K + \mathbb{L}^K] \underline{\Lambda} = 0$$

\Rightarrow **Symmetric linear system**

Main steps of the HDG algorithm

1. Construction of the global matrix \mathbf{M}

with $\mathbf{M} = \sum_{K \in \mathcal{T}_h} \left[-\mathbf{B}^K (\mathbf{A}^K)^{-1} \mathbf{C}^K + \mathbf{L}^K \right]$

for $K = 1$ to Nb_{tri} **do**

 Computation of matrices \mathbf{B}^K , $(\mathbf{A}^K)^{-1}$, \mathbf{C}^K and \mathbf{L}^K

 Construction of the corresponding section of \mathbf{M}

end for

Main steps of the HDG algorithm

-
1. Construction of the global matrix \mathbf{M}
 2. Construction of the right hand side \mathbf{S}
-

Main steps of the HDG algorithm

-
1. Construction of the global matrix \mathbf{M}
 2. Construction of the right hand side \mathbf{S}
 3. Resolution $\mathbf{M}\underline{\mathbf{\Lambda}} = \mathbf{S}$, with a direct solver (MUMPS) or hybrid solver (MaPhys)
-

Main steps of the HDG algorithm

-
1. Construction of the global matrix \mathbf{M}
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 3. Resolution $\mathbf{M}\underline{\mathbf{\Lambda}} = \mathbf{S}$, with a direct solver (MUMPS) or hybrid solver (MaPhys)
 4. Computation of the solutions of the initial problem
-

Main steps of the HDG algorithm

-
1. Construction of the global matrix $\underline{\mathbf{M}}$
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 3. Resolution $\underline{\mathbf{M}}\underline{\mathbf{\Lambda}} = \underline{\mathbf{S}}$, with a direct solver (MUMPS) or hybrid solver (MaPhys)
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for $K = 1$ to Nb_{tri} **do**
 Compute $\underline{\mathbf{W}}^K = -(\underline{\mathbf{A}}^K)^{-1}\underline{\mathbf{C}}^K\underline{\mathbf{\Lambda}}$
end for

Contents

Hybridizable Discontinuous Galerkin method

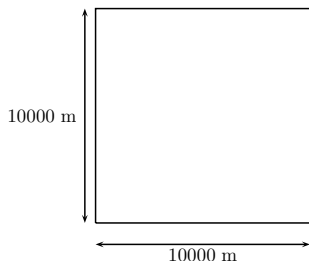
2D Numerical results : comparison of the two HDG formulations

Plane wave in an homogeneous medium

Anisotropic test case

3D numerical results : focus on the resolution part

Plane wave



Computational domain Ω
setting

► Physical parameters :

- $\rho = 2000 \text{ kg.m}^{-3}$
- $\lambda = 16 \text{ GPa}$
- $\mu = 8 \text{ GPa}$

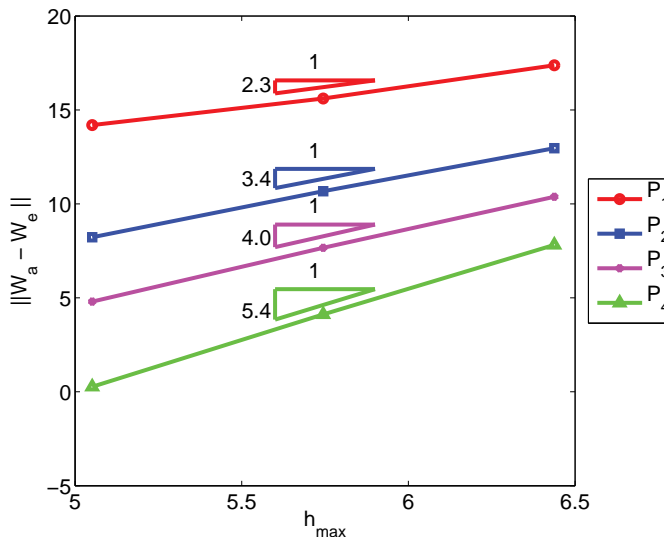
► Plane wave :

$$u = \nabla e^{i(k \cos \theta x + k \sin \theta y)}$$

where $k = \frac{\omega}{v_p}$

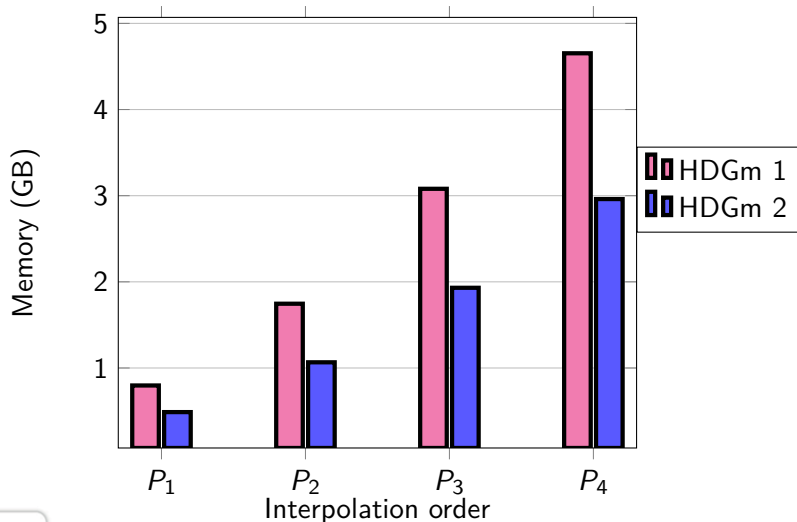
- $\theta = 0, v_p = 4000 \text{ m.s}^{-1}, \omega = 4\pi$
- Three meshes :
 - 3000 elements
 - 10000 elements
 - 45000 elements

Plane wave : Convergence order

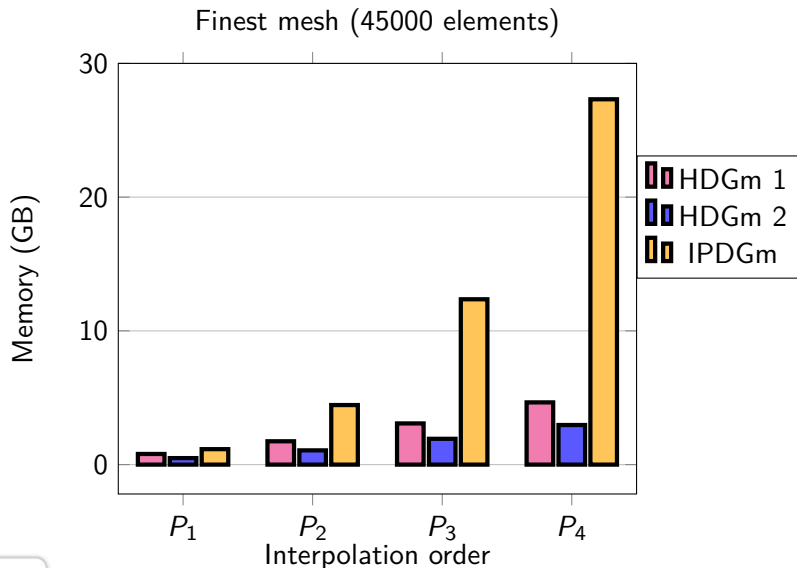


Plane wave : Memory consumption

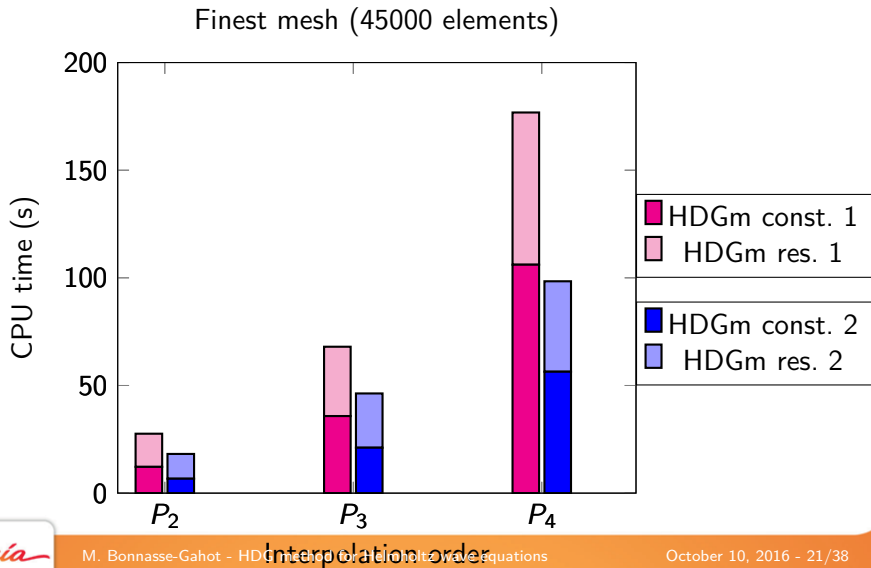
Finest mesh (45000 elements)



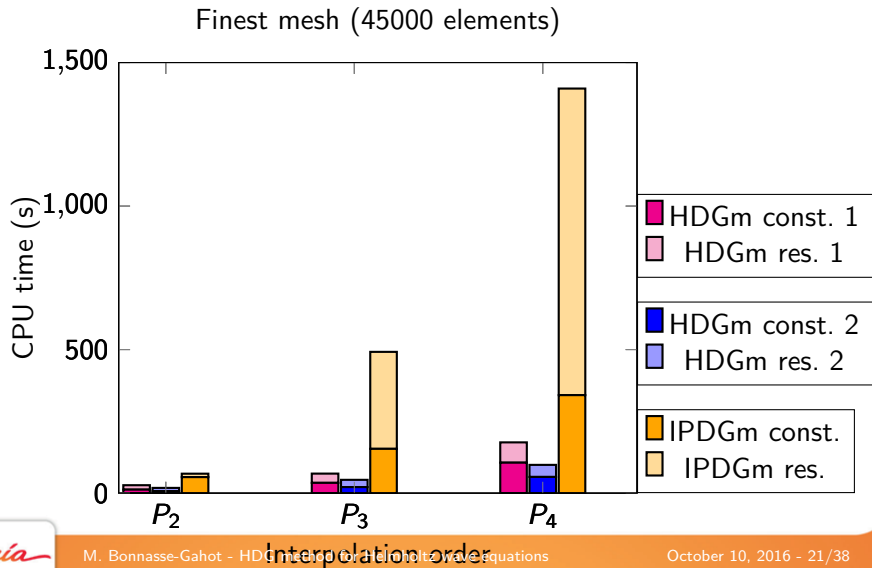
Plane wave : Memory consumption



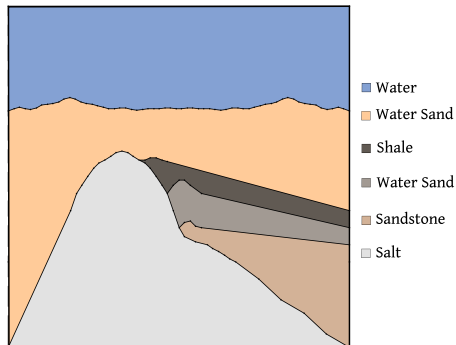
Plane wave : CPU time



Plane wave : CPU time

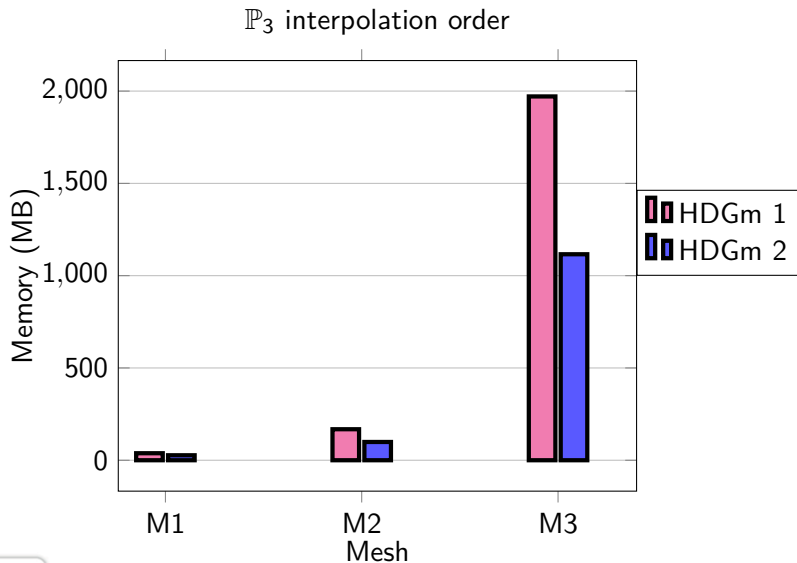


Anisotropic test case

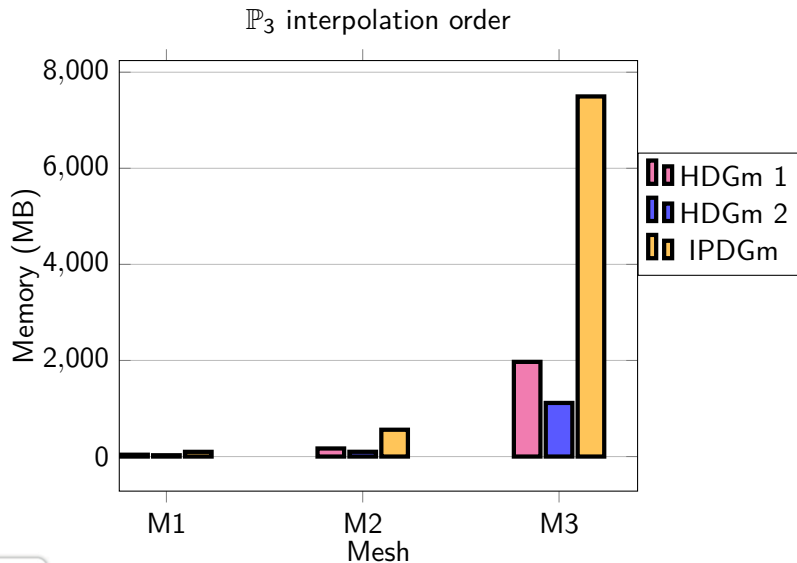


- ▶ Three meshes :
 - ▶ 600 elements
 - ▶ 3000 elements
 - ▶ 28000 elements

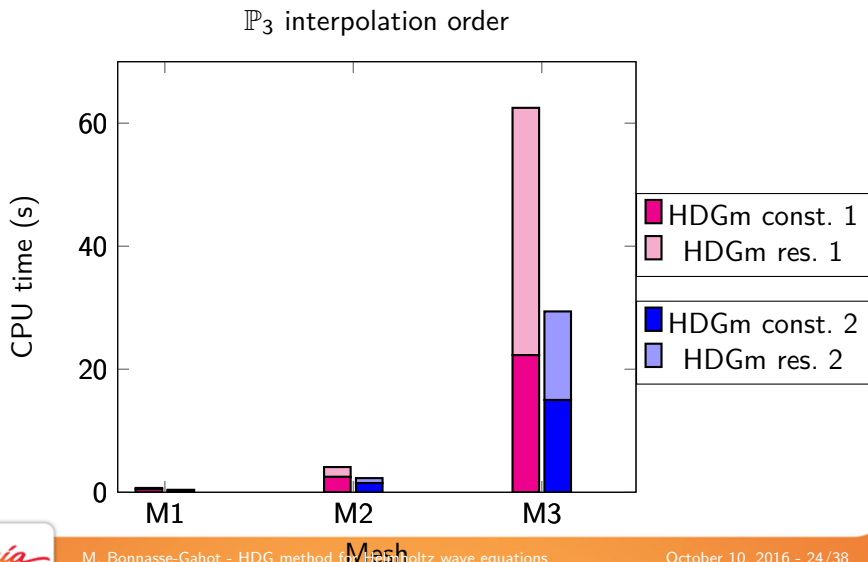
Anisotropic case : Memory consumption



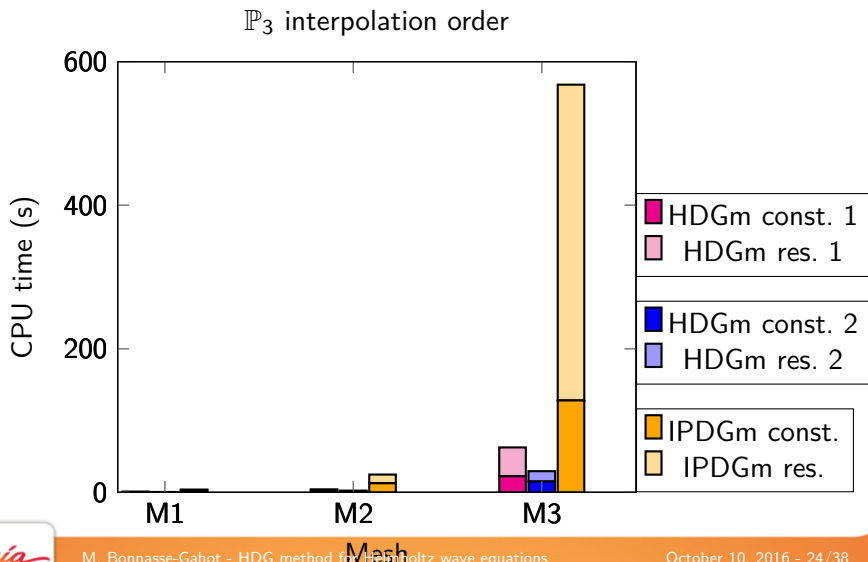
Anisotropic case : Memory consumption



Anisotropic case : CPU time (s)



Anisotropic case : CPU time (s)



Conclusion

- ▶ HDG method more efficient than classical DG methods for a same accuracy
 - ▶ Memory
 - ▶ Computational time

2D specific study of HDG formulation

- ▶ Anisotropic HDG algorithm without any additional computational cost
- ▶ Computational gain without loss of accuracy using p -adaptivity

Contents

Hybridizable Discontinuous Galerkin method

2D Numerical results : comparison of the two HDG formulations

3D numerical results : focus on the resolution part

3D plane wave in an homogeneous medium

3D geophysic test-case : Epati test-case

Main steps of the HDG algorithm

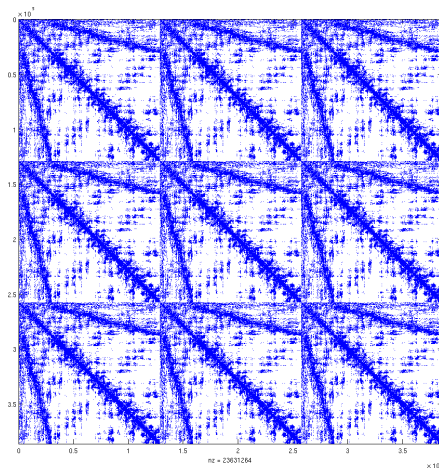
-
1. Construction of the global matrix \mathbf{M}
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 4. Computation of the solutions of the initial problem
-

Main steps of the HDG algorithm

-
1. Construction of the global matrix \mathbf{M}
 2. Construction of the right hand side \mathbf{S}
 3. **Resolution $\mathbf{M}\underline{\mathbf{A}} = \mathbf{S}$, with a direct solver (MUMPS) or hybrid solver (MaPhys)**
 4. Computation of the solutions of the initial problem
-

MaPhys Vs MUMPS

Pattern of the HDG global matrix for \mathbb{P}_1 interpolation and for a 3D mesh composed of 21 000 elements

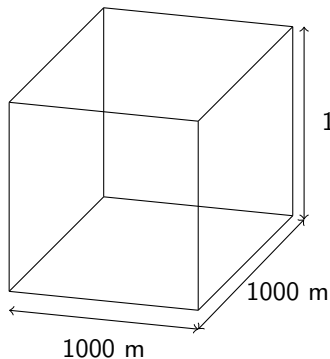


MaPhys Vs MUMPS

Software packages for solving systems of linear equations $Ax = b$, where A is a sparse matrix

- ▶ MUMPS (MULTifrontal Massively Parallel sparse direct Solver) :
 - ▶ Direct factorization $A = LU$ or $A = LDL^T$
 - ▶ Multifrontal approach
- ▶ MaPhys (Massively Parallel Hybrid Solver) :
 - ▶ Direct and iterative methods
 - ▶ non-overlapping algebraic domain decomposition method (Schur complement method)
 - ▶ resolution of each local problem thanks to direct solver such as MUMPS or PaStiX.

3D plane wave in an homogeneous medium



Configuration of the computational domain Ω .

► Physical parameters :

- $\rho = 1 \text{ kg.m}^{-3}$
- $\lambda = 16 \text{ GPa}$
- $\mu = 8 \text{ GPa}$

► Plane wave :

$$u = \nabla e^{i(k_x x + k_y y + k_z z)}$$

where $k_x = \frac{\omega}{v_p} \cos \theta_0 \cos \theta_1$,

$k_y = \frac{\omega}{v_p} \sin \theta_0 \cos \theta_1$, and

$k_z = \frac{\omega}{v_p} \sin \theta_1$

- $\omega = 2\pi f$, $f = 8 \text{ Hz}$
- $\theta_0 = 30^\circ$, $\theta_1 = 0^\circ$
- Mesh composed of 21 000 elements

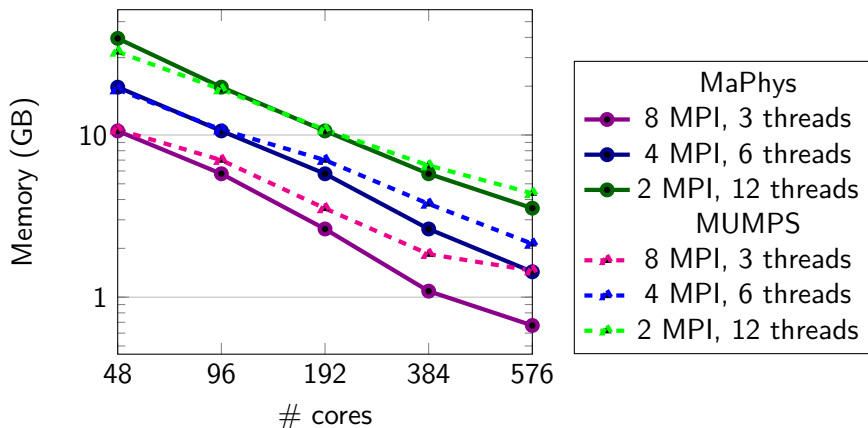
Cluster configuration

Features of the nodes :

- ▶ 2 Dodeca-core Haswell Intel Xeon E5-2680
- ▶ Frequency : 2,5 GHz
- ▶ RAM : 128 Go
- ▶ Storage : 500 Go
- ▶ Infiniband QDR TrueScale : 40Gb/s
- ▶ Ethernet : 1Gb/s

3D Plane wave : Memory consumption

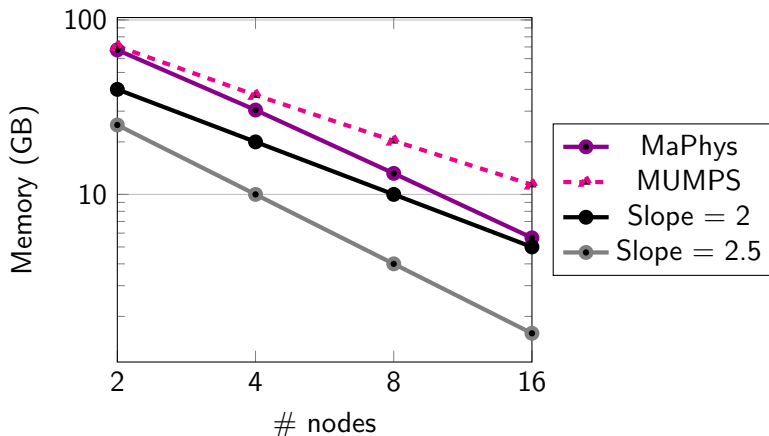
Maximum local memory for HDG- \mathbb{P}_3 method



(matrix order = 1 287 360, # nz=298 598 400)

3D Plane wave : Memory consumption

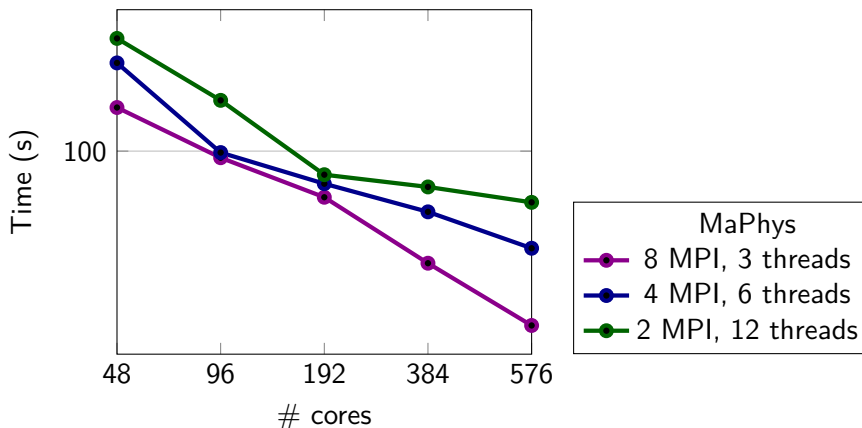
Average memory for one node (8 MPI by node and 3 threads by MPI)



(matrix order = 1 287 360, # nz=298 598 400)

3D Plane wave : Execution time

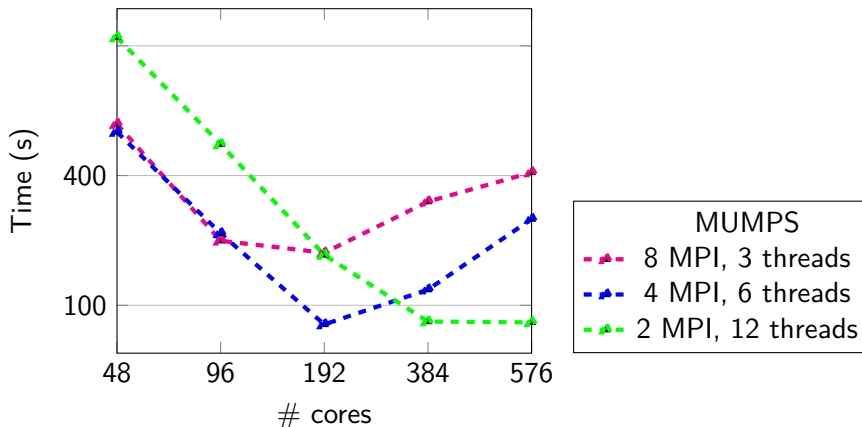
Execution time for the resolution of the HDG- \mathbb{P}_3 system



(matrix order = 1 287 360, # nz=298 598 400)

3D Plane wave : Execution time

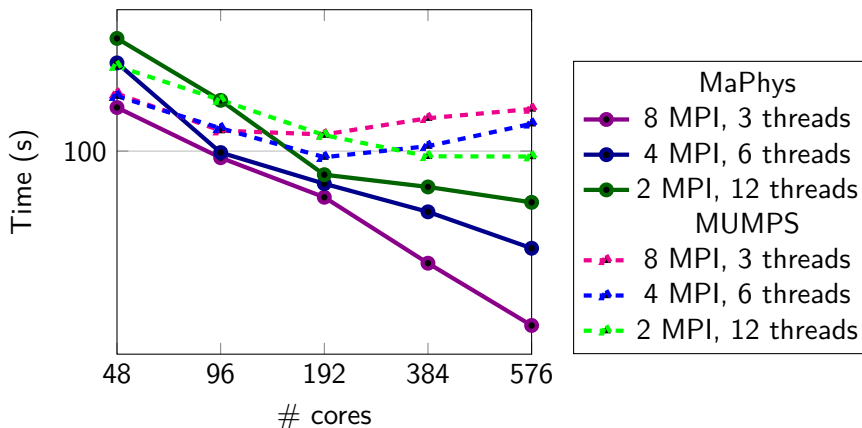
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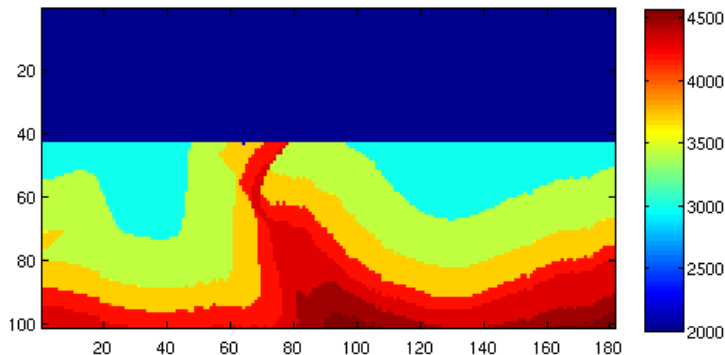
3D Plane wave : Execution time

Execution time for the resolution of the HDG- \mathbb{P}_3 system



(matrix order = 1 287 360, # nz=298 598 400)

Epati test-case

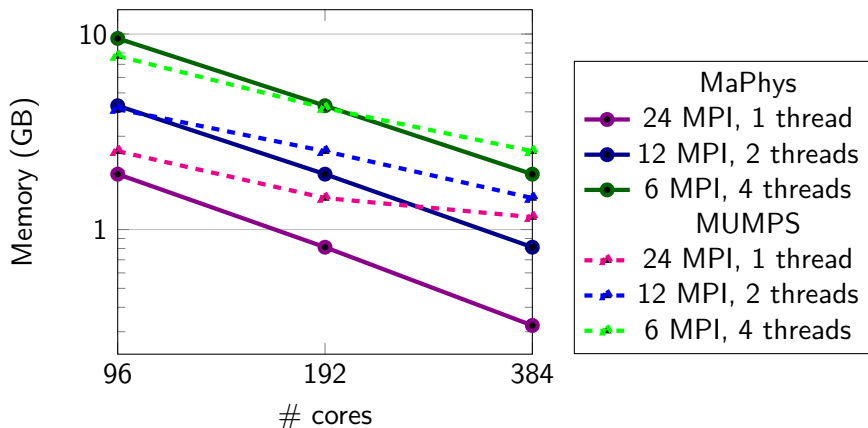


V_p -velocity model (m.s^{-1}), vertical section at $y = 700$ m

Mesh composed of 25 000 tetrahedrons

Epati test-case : Memory consumption

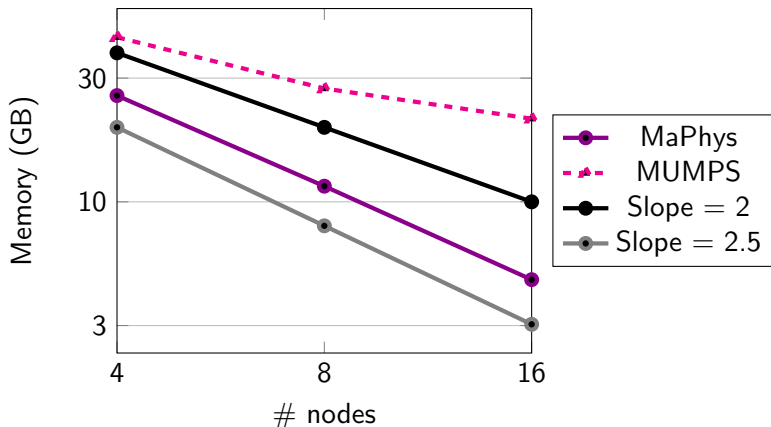
Maximum local memory for HDG- \mathbb{P}_3 method



(matrix order = 1 600 740, # nz=365 385 600)

Epati test-case : Memory consumption

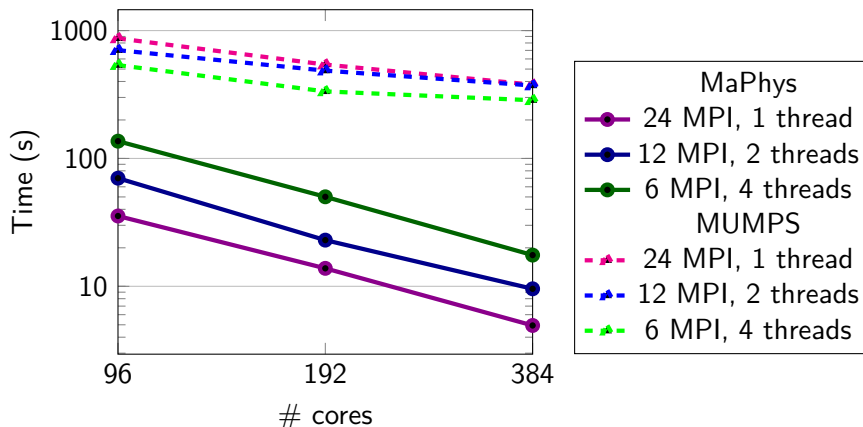
Average memory for one node (24 MPI by node and 1 thread by MPI)



(matrix order = 1 287 360, # nz=365 385 600)

Epati test-case : Execution time

Execution time for the resolution of the HDG- \mathbb{P}_3 system



(matrix order = 1 287 360, # nz=365 385 600)

Conclusion-Perspectives

- ▶ more detailed analysis of the comparison between MUMPS and MaPhys
 - ▶ comparison for the symmetric HDG formulation
- ▶ comparison to PaStiX solver
- ▶ extension to elasto-acoustic case
- ▶ study of the stabilization parameter τ for the 3D case
- ▶ call for projects PRACE to test bigger test-cases

Thank you !

